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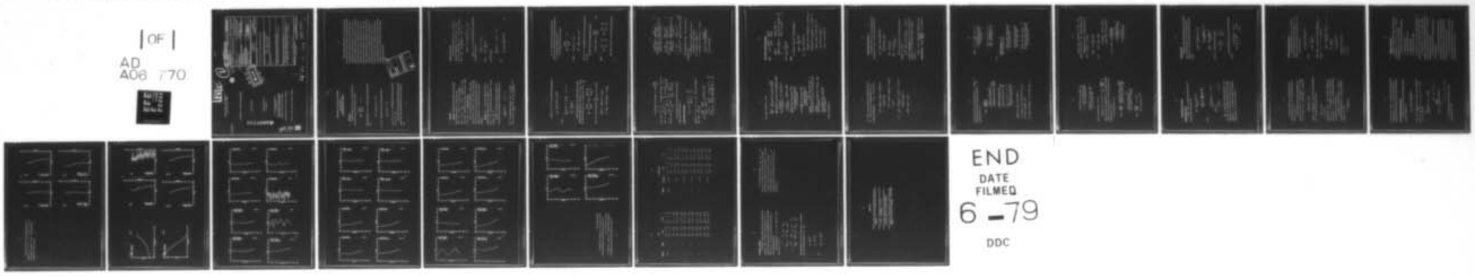

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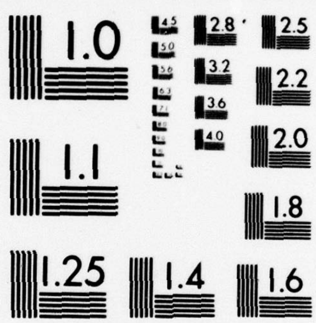

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# INVESTIGATION OF A MULTIPLE TIME SERIES MODEL

By David Tritchler

Technical Report No. N-4

February 1979


Texas A & M Research Foundation
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"Multiple Time Series Modeling and Time Series Theoretic Statistical Methods"

Sponsored by the Office of Naval Research

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We show how modern techniques of multiple time series analysis can be used to determine if two time series are related by the model:
$Y(t) = \gamma_0 X(t) + \gamma_1 X(t-1) + \eta(t)$, $X(t) + \alpha X(t-1) = \varepsilon(t)$.

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# INVESTIGATION OF A MULTIPLE TIME SERIES MODEL

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## 1. Introduction

The subject of this project is the so-called regression time series model: i.e. the two dimensional time series

$$\underset{\sim}{Z}(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} , \quad t \in Z ,$$

where the time series $Y(\cdot)$ is linearly related to the time series $X(\cdot)$ :

$$Y(t) = \gamma_0 X(t) + \gamma_1 X(t-1) + \eta(t) ,$$

and $X(\cdot)$ satisfies the first order autoregressive model

$$X(t) + \alpha X(t-1) = \epsilon(t) ,$$

and the $\epsilon(\cdot)$ and $\eta(\cdot)$ are independent white noise processes with variances $\sigma_\epsilon^2$ , $\sigma_n^2$ .

Thus we show how modern techniques of multiple time series analysis can be used to determine if two time series are related as above.





Chapter 2 defines multiple time series, covariance stationary time series, the autocovariance function, the multiple spectral density, the autoregressive representation, and the periodic autoregressive representation of a multiple time series. The time series $\underset{\sim}{Z}(t)$ is expressed as a multiple autoregression and conditions for stationarity, the autocovariance function, and the spectral density and some of its properties are derived. Chapter 3 defines coherence, phase, and gain and derives these quantities for the specific time series $\underset{\sim}{Z}(t)$ . Chapter 4 addresses the problem of estimating the spectral density of a stationary time series, defining the sample spectral density, the kernel method, the stationary autoregressive method, and the periodic autoregressive method. Chapter 5 presents the results of a study comparing the various multiple spectral estimators and makes conclusions on how they can be used to determine if two time series satisfy the regression model.

## 2. The Model

**Definition:** Multiple time series analysis is concerned with finding relationships among $d$ univariate time series $\{X_1(t), t \in Z\}, \ldots, \{X_d(t), t \in Z\}$ given finite realizations $\{X_1(t), 1 \le t \le T\}, \ldots, \{X_d(t), 1 \le t \le T\}$. Grouping the $d$ series into a series of $d$-dimensional random vectors $\underset{\sim}{X}(t) = (X_1(t), \ldots, X_d(t))^T$, we call $\{\underset{\sim}{X}(t), t \in Z\}$ a multiple time series.

Since we are interested in the probability law of time series, usually assumed to be Gaussian, we wish to know its covariance kernel. To achieve this in practice an assumption must be made to reduce the number of parameters to be estimated, that of weak (covariance) stationarity.

**Definition:** $\underset{\sim}{X}(\cdot)$ is a covariance stationary time series (CSTS) with autocovariance function $R(v) = (R_{jk}(v))$, $v \in Z$ if $\forall j,k = 1,\ldots,d$, $\exists$ a real valued function on the integers, $R_{j,k}(v) = \mathrm{Cov}(X_j(t), X_k(t+v))$.

In addition, if a mixing type assumption is satisfied we can use the powerful tool of multiple spectral density estimation, i.e. if

$$\sum_{v=-\infty}^{\infty} \left|R_{jk}(v)\right| < \infty \qquad j,k = 1,\ldots,d$$

then $\exists$ the multiple spectral density of $\underset{\sim}{X}(\cdot)$, $f(\omega) = (f_{jk}(\omega))$ $x \in [-\pi, \pi] \ni R_{jk}(v) = \int_{-\pi}^{\pi} f_{jk}(\omega) e^{iv\omega} d\omega$ and $f_{jk}(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} R_{jk}(v) e^{-iv\omega}$.



**Theorem** (Parzen (1976))

A CSTS with multiple spectral density $f(\cdot)$ has an autoregressive representation if $\exists \lambda_1, \lambda_2 > 0 \ni f(\omega) - \lambda_1 I$ and $\lambda_2 I - f(\omega)$ are positive definite, $\forall \omega$. Then $\exists$ $d \times d$ matrices $A(0) = I$, $A(1), \ldots, \underset{\sim}{\Sigma} \ni \sum_{j=0}^{\infty} A(j) \underset{\sim}{X}(t-j) = \underset{\sim}{\varepsilon}(t)$, $t \in Z$

$$E[\underset{\sim}{\varepsilon}(t)] = 0 \quad \text{and} \quad E[\underset{\sim}{\varepsilon}(t) \underset{\sim}{\varepsilon}^T(t+v)] = \delta_{v,o} \underset{\sim}{\Sigma} .$$

Further, $\underset{\sim}{X}(\cdot)$ has a stationary autoregressive representation if, in addition to the above,

$$\det(G(z)) = 0 \rightarrow |z| > 1$$

where $G(z) = \sum_{j=0}^{\infty} A(j) z^j$.

Then we may write

$$\sum_{j=0}^{\infty} A(j) R(j-v) = \delta_{v,o} \underset{\sim}{\Sigma}, \qquad v \ge 0$$

and

$$f(\omega) = \frac{1}{2\pi} G^{-1}(e^{i\omega}) \underset{\sim}{\Sigma} G^{-*}(e^{i\omega}) . \tag{2.1}$$

In practice, we use a $p^{th}$ order autoregressive approximation

$$f_p(z) = \frac{1}{2\pi} G_p^{-1}(e^{i\omega}) \Sigma_p \, G_p^{-*}(e^{i\omega})$$

where

$$\sum_{j=0}^{p} A_p(j) R(j-v) = \delta_{v,o} \Sigma_p \, , \quad v = 0, \ldots, p$$

$$G_p(z) = \sum_{j=0}^{p} A_p(j) z^j$$

Another representation of a time series is that of a periodic autoregression (Pagano (1976)): given $\underset{\sim}{X}(\cdot)$, $p$, $A_p(1), \ldots, A_p(p)$, $\Sigma_p$ form the scalar series $Y(\cdot)$ by $X_j(t) = Y((t-1)d + j)$, i.e.

$$\underset{\sim}{X}(1) = \begin{pmatrix} Y(1) \\ \vdots \\ Y(d) \end{pmatrix}, \quad \underset{\sim}{X}(2) = \begin{pmatrix} Y(d+1) \\ \vdots \\ Y(2d) \end{pmatrix}, \ \ldots .$$

Then $Y(\cdot)$ can be represented by $\sum_{j=0}^{p_t} \alpha_t(j) Y(t-j) = \eta(t)$ where

$$E(\eta(t)) = 0 \, , \quad E[\eta(t) \, \eta(t+v)] = \delta_{v,o} \sigma_t^2 \, ,$$

$$p_t = p_{t+kd} \, , \quad \alpha_t(j) = \alpha_{t+kd}(j) \, , \quad \sigma_t^2 = \sigma_{t+kd}^2$$



So $Y(\cdot)$ is like a scalar autoregression but the order, coefficients, and residual variances are the same for like channels in $\underset{\sim}{X}$ and different for different channels.

The 2-dimensional time series of interest is

$$\underset{\sim}{Z}(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \quad t \in Z$$

where $\quad X(t) + \alpha X(t-1) = \epsilon(t)$

$$Y(t) = \gamma_0 X(t) + \gamma_1 X(t-1) + \eta(t)$$

, and $\epsilon(\cdot)$, $\eta(\cdot)$ are independent white noise processes with variance $\sigma_\epsilon^2$, $\sigma_\eta^2$.

This may be written as the multiple autoregression

$$A(0) \underset{\sim}{Z}(t) + A(1) \underset{\sim}{Z}(t-1) = \begin{pmatrix} \epsilon(t) \\ \eta(t) \end{pmatrix} \qquad (2.2)$$

where

$$A(0) = \begin{bmatrix} 1 & 0 \\ -\gamma_0 & 1 \end{bmatrix} \qquad A(1) = \begin{bmatrix} \alpha & 0 \\ -\gamma_1 & 0 \end{bmatrix} .$$

So that $A(0)$ will equal $I$ we may rewrite the autoregression as

$$A(0)^{-1}A(0)\,\underset{\sim}{Z}(t) + A(0)^{-1}A(1)\,\underset{\sim}{Z}(t-1) = A(0)^{-1}\begin{pmatrix}\epsilon(t)\\ \eta(t)\end{pmatrix}$$

where $A(0)^{-1} = \begin{bmatrix} 1 & 0 \\ \gamma_0 & 1 \end{bmatrix}$. This yields $\underset{\sim}{Z}(t) + A\,\underset{\sim}{Z}(t-1) = \underset{\sim}{\lambda}(t)$ (2.3)

where $A = A(0)^{-1}A(1) = \begin{bmatrix} \alpha & 0 \\ \alpha\gamma_0 - \gamma_1 & 0 \end{bmatrix}$ and $\underset{\sim}{\lambda}(t) = \begin{pmatrix} \epsilon(t) \\ \gamma_0\epsilon(t) + \eta(t) \end{pmatrix}$.

Here $\Sigma = A(0)^{-1}\begin{bmatrix} \sigma_\epsilon^2 & 0 \\ 0 & \sigma_\eta^2 \end{bmatrix} A(0)^{-T} = \begin{bmatrix} \sigma_\epsilon^2 & \sigma_\epsilon^2\gamma_0 \\ \gamma_0\sigma_\epsilon^2 & \gamma_0^2\sigma_\epsilon^2 + \sigma_\eta^2 \end{bmatrix}$

If the representation (2.3) is stationary, we may obtain the multiple spectral density from (2.1).

**Spectral Density of the Model**

$$G(z) = I + Az = \begin{bmatrix} \alpha z + 1 & 0 \\ z(\alpha\gamma_0 - \gamma_1) & 1 \end{bmatrix}$$

$\det(G(z)) = 0 \to \alpha z + 1 = 0 \to z = -\frac{1}{\alpha}$. Hence (2.3) is stationary when $|\alpha| < 1$. In this case

$$f_z(\omega) = \begin{bmatrix} f_{xx}(\omega) & f_{xy}(\omega) \\ f_{yx}(\omega) & f_{yy}(\omega) \end{bmatrix} = \frac{1}{2\pi} G^{-1}(e^{i\omega})\,\Sigma\, G^{-*}(e^{i\omega})$$

$$= \begin{bmatrix} \alpha e^{i\omega}+1 & 1 \\ e^{i\omega}(\alpha\gamma_0 - \gamma_1) & 0 \end{bmatrix}\begin{bmatrix} \sigma_\epsilon^2 & \sigma^2\gamma_0 \\ \gamma_0\sigma_\epsilon^2 & \gamma_0^2\sigma_\epsilon^2 + \sigma_\eta^2 \end{bmatrix}\begin{bmatrix} \alpha e^{-i\omega}+1 & 1 \\ e^{-i\omega}(\alpha\gamma_0 - \gamma_1) & 0 \end{bmatrix}^{-T}$$



$$= \frac{1}{2\pi}\begin{bmatrix} \dfrac{\sigma_\epsilon^2}{(\alpha e^{i\omega}+1)(\alpha e^{-i\omega}+1)} & \dfrac{\sigma_\epsilon^2}{(\alpha e^{i\omega+1})(\alpha e^{-i\omega}+1)} \\ \dfrac{\sigma_\epsilon^2(\gamma_0 + \gamma_1 e^{i\omega})}{(\alpha e^{i\omega}+1)(\alpha e^{-i\omega}+1)} & \dfrac{\sigma_\epsilon^2(\gamma_0+\gamma_1 e^{i\omega})(\gamma_0+\gamma_1 e^{-i\omega})}{(\alpha e^{i\omega}+1)(e^{-i\omega}+1)} + \sigma_\eta^2 \end{bmatrix}$$

$$= \frac{1}{2\pi}\begin{bmatrix} \dfrac{\sigma_\epsilon^2}{\alpha^2 + 2\alpha\cos(\omega)+1} & \dfrac{\sigma_\epsilon^2(\gamma_0 + \gamma_1 e^{-i\omega})}{\alpha^2 + 2\alpha\cos(\omega)+1} \\ \dfrac{\sigma_\epsilon^2(\gamma_0 + \gamma_1 e^{i\omega})}{\alpha^2 + 2\alpha\cos\omega + 1} & \dfrac{\sigma_\epsilon^2(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)}{\alpha^2 + 2\alpha\cos(\omega)+1} + \sigma_\eta^2 \end{bmatrix}$$

$$= \begin{bmatrix} f_{xx}(\omega) & f_{xx}(\omega)(\gamma_0 + \gamma_1 e^{-i\omega}) \\ f_{xx}(\omega)(\gamma_0 + \gamma_1 e^{i\omega}) & f_{xx}(\omega)(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2) + \frac{\sigma_\eta^2}{2\pi} \end{bmatrix}$$

Some observations about $f(\omega)$, $\omega \in (0,\pi)$ (the interval our graphs portray) are:

(i) $\sigma_\epsilon^2\uparrow \Rightarrow f_{xx}(\omega)\uparrow$ and $f_{yy}(\omega)\uparrow$ where the level, but not the shape of the curve is changed.

(ii) $\alpha > 0 \Rightarrow f_{xx}(\omega)$ is monotone $\uparrow$ in $\omega$.

$\alpha < 0 \Rightarrow f_{xx}(\omega)$ is monotone $\downarrow$ in $\omega$. This since

$$\frac{\partial}{\partial\pi} f_{xx}(\omega) = \frac{1}{2\pi}\,\frac{2\alpha\sin(\omega)\,\sigma_\epsilon^2}{(\alpha^2 + 2\alpha\cos(\omega)+1)^2},$$

which is of constant sign.

(iii) $f_{yy}(\omega)$ is monotone in $\omega$, since

$$\frac{\partial f_{yy}(\omega)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[f_{xx}(\omega)(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2) + \frac{\sigma_\eta^2}{2\pi}\right]$$

$$= \left(\frac{\partial}{\partial \omega} f_{xx}(\omega)\right)(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2) + f_{xx}(\omega)\left(2\gamma_1\gamma_0(-\sin(\omega))\right)$$

$$= \frac{1}{2\pi}\,\frac{2\alpha\sin(\omega)\,\sigma_\epsilon^2}{\alpha^2 + 2\alpha\cos(\omega) + 1}\,(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)$$

$$+ \frac{\sigma_\epsilon^2}{\alpha^2 + 2\alpha\cos(\omega) + 1}\cdot\frac{1}{2\pi}\left(2\gamma_1\gamma_0(-\sin(\omega))\right)$$

$$= \frac{\sigma_\epsilon^2}{2\pi}\,\frac{\begin{pmatrix}2\alpha\sin(\omega)(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)\\ + 2\gamma_1\gamma_0(-\sin(\omega))(\alpha^2 + 2\alpha\cos(\omega) + 1)\end{pmatrix}}{(\alpha^2 + 2\alpha\cos(\omega) + 1)^2}$$

$$= \frac{\sigma_\epsilon^2\sin(\omega)\begin{pmatrix}2\alpha(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)\\ - 2\gamma_1\gamma_0(\alpha^2 + 2\alpha\cos(\omega) + 1\end{pmatrix}}{2\pi(\alpha^2 + 2\alpha\cos(\omega) + 1)^2}$$

$$= \frac{\sigma_\epsilon^2\sin(\omega)}{2\pi(\alpha^2 + 2\alpha\cos(\omega) + 1)^2}\begin{pmatrix}2\alpha\gamma_0^2 + 4\alpha\gamma_1\gamma_0\cos(\omega) + 2\alpha\gamma_1^2\\ - 2\gamma_1\gamma_0\alpha^2 - 4\alpha\gamma_1\gamma_0\cos(\omega) - 2\gamma_1\gamma_0\end{pmatrix}$$

$$= \frac{\sigma_\epsilon^2\sin(\omega)}{2\pi(\alpha^2 + 2\alpha\cos(\omega) + 1)^2}\left(2\gamma_0^2\alpha + 2\alpha\gamma_1^2 - 2\alpha^2\gamma_1\gamma_0 - 2\gamma_1\gamma_0\right)$$

$$= \frac{2\sigma_\epsilon^2\sin(\omega)\left(\alpha(\gamma_0^2 + \gamma_1^2) - \gamma_1\gamma_0(\alpha^2 + 1)\right)}{2\pi(\alpha^2 + 2\alpha\cos(\omega) + 1)^2},$$

which is of constant sign.



**Autocovariance Function of the Model**

**Note:** $R_Z(v) = \begin{bmatrix} R_{xx}(v) & R_{xy}(v) \\ R_{yx}(v) & R_{yy}(v) \end{bmatrix}$

$$= \begin{bmatrix} R_{xx}(v) & \gamma_0 R_{xx}(v) + \gamma_1 R_{xx}(v-1) \\ \gamma_0 R_{xx}(v) + \gamma_1 R_{xx}(v+1) & R_{xx}(v)(\gamma_0^2 + \gamma_1^2) + \gamma_0\gamma_1(R_{xx}(v-1) + R_{xx}(v+1)) + \delta_{0,v}\sigma_\eta^2 \end{bmatrix}$$

where $R_{xx}(v) = \dfrac{\sigma_\epsilon^2}{1-\alpha^2}(-\alpha)^{|v|}$

**Proof**

(i) Since the X-series is a 1st order autoregression,

$$R_{xx}(v) = \frac{\sigma_\epsilon^2}{1-\alpha^2}(-\alpha)^{|v|}$$

(ii) $R_{yy}(v) = E[Y(t)\,Y(t+v)]$

$$= E[(\gamma_0 X(t) + \gamma_1 X(t-1) + \eta(t))(\gamma_0 X(t+v) + \gamma_1 X(t+v-1) + \eta(t+v$$

$$= E[\gamma_0^2 X(t)X(t+v) + \gamma_0\gamma_1 X(t)X(t+v-1) + \gamma_0 X(t)\eta(t+v)$$

$$+ \gamma_1\gamma_0 X(t-1)X(t+v) + \gamma_1^2 X(t-1)X(t+v-1)$$

$$+ \gamma_1 X(t-1)\eta(t+v) + \eta(t)\gamma_0 X(t+v)$$

$$+ \gamma_1 X(t+v-1)\eta(t) + \eta(t)\eta(t+v)]$$

$|\alpha| < 1 \Rightarrow X(\cdot)$ has a moving average representation in terms of $\epsilon(\cdot)$. Then $\epsilon(\cdot)$ and $\eta(\cdot)$ independent $\Rightarrow X(\cdot)$ and $\eta(\cdot)$ independent $\Rightarrow E[X_{(j)}\eta_{(k)}] = 0 \quad \forall_{j,k}$. Therefore we have

$$R_{yy}(v) = \gamma_0^2 R_{xx}(v) + \gamma_0\gamma_1 R_{xx}(v-1) + \gamma_0\gamma_1 R_{xx}(v+1)$$
$$+ \gamma_1^2 R_{xx}(v) + R_\eta(v)$$
$$= (\gamma_0^2 + \gamma_1^2) R_{xx}(v) + \gamma_0\gamma_1(R_{xx}(v-1) + R_{xx}(v+1))$$
$$+ \delta_{o,v}\sigma_\eta^2$$

(iii) $$R_{xy}(v) = E[X(t)(\gamma_0 X(t+v) + \gamma_1 X(t+v-1) + \eta(t+v)]$$
$$= \gamma_0 R_{xx}(v) + \gamma_1 R_{xx}(v-1) \ .$$

Finally, $R_{yx}(v) = R_{xy}(-v)$ .



3. <u>Quantities Derived from Spectra</u>

A univariate time series $X_j(\cdot)$ may be represented, to any desired degree of accuracy, by a linear combination of sinusoids,

$$X_j(t) = \sum_k [\alpha_j(k)\cos(t\omega_k) + \beta_j(k)\sin(t\omega_k)]$$
$$= \sum_k \rho_j(k)\cos(t\omega_k - \varphi_j(k)) \ ,$$

$$\rho_j^2(k) = \alpha_j^2(k) + \beta_j^2(k)$$

$$\varphi_j(k) = \tan^{-1}\left(\beta_j(k)/\alpha_j(k)\right) \ .$$

We call $\rho_j(k)\cos(t\omega_k - \varphi_j(k))$ the frequency component of frequency $\omega_k$ of $X_j(\cdot)$ .

Note that $R_{jj}(0) = \mathrm{Var}\, X_j(t) = \int_{-\pi}^{\pi} f_{jj}(\omega)\, d\omega$ , therefore we interpret the <u>power spectrum</u> $f_{jj}(\omega)$ as the measure of amount of variability in $X_j(\cdot)$ contributed by the frequency component of frequency $\omega$ .

Similarly, $R_{jk}(0) = \mathrm{cov}(X_j(t), X_k(t)) = \int_{-\pi}^{\pi} f_{jk}(\omega)\, d\omega$ and the <u>squared coherence</u> $0 \le W_{jk}(\omega) = \dfrac{|f_{jk}(\omega)|^2}{f_{jj}(\omega) f_{kk}(\omega)} \le 1$ is a standardized

measure of the amount of variability between series $j$ and series $k$ contributed by their frequency components of frequency $\omega$. For our particular series $Z(\cdot)$,

$$W_{yx}(\omega) = \frac{|f_{xy}(\omega)|^2}{f_{xx}(\omega) f_{yy}(\omega)}$$

$$= \frac{f_{xx}^2(\omega)(\gamma_0 + \gamma_1 e^{-i\omega})(\gamma_0 + \gamma_1 e^{i\omega})}{f_{xx}(\omega)\left[f_{xx}(\omega)(\gamma_0^2 + 2\gamma_1\gamma_0 \cos(\omega) + \gamma_1^2) + \frac{\sigma_\eta^2}{2\pi}\right]}$$

$$= \frac{f_{xx}(\omega)(\gamma_0^2 + 2\gamma_1\gamma_0 \cos(\omega) + \gamma_1^2)}{f_{xx}(\omega)(\gamma_0^2 + 2\gamma_1\gamma_0 \cos(\omega) + \gamma_1^2) + \frac{\sigma_\eta^2}{2\pi}}$$

$$= \frac{1}{1 + \frac{\sigma_\eta^2}{2\pi f_{xx}(\omega)[\gamma_0^2 + 2\gamma_1\gamma_0 \cos(\omega) + \gamma_1^2]}}$$

Properties of $W_{yx}(\omega)$ for our model are:

(i) The value of $W_{yx}(\omega) \uparrow$ uniformly $(\omega \in (0,\pi))$ as $\sigma_\eta^2 \uparrow$ and $\downarrow$ uniformly as $\sigma_\eta^2 \uparrow$.

(ii) $W_{yx}(\omega)$ is monotonic in $\omega$ for $\omega \in (0,\pi)$, since



$$W_{yx}(\omega) = \frac{1}{1 + \frac{\sigma_\eta^2}{\frac{2\pi\sigma_\epsilon^2(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)}{\alpha^2 + 2\alpha\cos(\omega) + 1}}}$$

$$= \frac{1}{1 + \frac{\sigma_\eta^2}{2\pi\sigma_\epsilon^2} \frac{\alpha^2 + 2\alpha\cos(\omega) + 1}{\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2}}$$

and

$$\frac{\partial}{\partial\omega}\left(\frac{\alpha^2 + 2\alpha\cos(\omega) + 1}{\gamma_0^2 + 2\alpha_1\gamma_0\cos(\omega) + \gamma_1^2}\right)$$

$$= \frac{(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)(2\alpha)(-\sin(\omega))}{(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)^2}$$

$$- \frac{(\alpha^2 + 2\alpha\cos(\omega) + 1)(2\gamma_1\gamma_0)(-\sin(\omega))}{(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)^2}$$

$$= \frac{\left[\begin{array}{l}(\alpha^2 + 2\alpha\cos(\omega) + 1)(2\gamma_1\gamma_0\sin(\omega)) \\ - (\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)(2\alpha\sin(\omega))\end{array}\right]}{(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)^2}$$

$$= \frac{2\sin(\omega)\left[\alpha^2\gamma_1\gamma_0 + 2\alpha\gamma_1\gamma_0\cos(\omega) + \gamma_1\gamma_0 - \alpha\gamma_0^2 - 2\gamma_1\gamma_0\alpha\cos(\omega) - \gamma_1^2\alpha\right]}{(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)^2}$$

$$= \frac{2\sin(\omega)\left[\alpha^2\gamma_1\gamma_0 + \gamma_1\gamma_0 - \alpha(\gamma_0^2 + \gamma_1^2)\right]}{(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)^2}$$

which is of constant sign for $\omega \in (0, \pi)$

The gain of series $j$ given series $k$ $g_{jk}(\omega) = \dfrac{|f_{jk}(\omega)|}{f_{kk}(\omega)}$ is a measure of the ratio of the amplitudes of the $\omega$ frequency components of series $k$ and of the fitted series $j$ formed by regressing series $j$ on lagged values of series $k$.

For our particular series $Z(\cdot)$,

$$g_{xy}(\omega) = \frac{|f_{xy}(\omega)|}{f_{yy}(\omega)} = \frac{\left[f_{xx}^2(\omega)(\gamma_0 + \gamma_1 e^{-i\omega})(\gamma_0 + \gamma_1 e^{i\omega})\right]^{1/2}}{f_{xx}(\omega)(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2) + \frac{\sigma_\eta^2}{2\pi}}$$

$$= \frac{(\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)^{1/2}}{\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2 + \sigma_\eta^2/f_{xx}(\omega)}$$

$$g_{yx}(\omega) = \frac{|f_{yx}(\omega)|}{f_{xx}(\omega)} = |\gamma_0 + \gamma_1 e^{i\omega}|$$

$$= (\gamma_0^2 + 2\gamma_1\gamma_0\cos(\omega) + \gamma_1^2)^{1/2}$$



Note that $g_{yx}(\omega)$ is monotonic for $\omega \in (0, \pi)$. Also, $g_{xy}(\omega)\uparrow$ uniformly as $\sigma_\varepsilon^2\uparrow$, while $g_{xy}(\omega)\downarrow$ uniformly as $\sigma_\eta^2\uparrow$.

The phase angle between the $\omega$ frequency components of series $j$ and series $k$ is given by

$$\varphi_{jk}(\omega) = -\tan^{-1}\left(-q_{jk}(\omega)/C_{jk}(\omega)\right)$$

where

$$f_{jk}(\omega) = C_{jk}(\omega) - i\,q_{jk}(\omega)$$

$$= \text{co-spectrum} - i \text{ quadrature spectrum}$$

$$= |f_{jk}(\omega)|\, e^{-i\varphi_{jk}(\omega)} \quad \text{in polar coordinates.}$$

For $Z(\cdot)$, $\varphi(\omega) = \arctan\left(\dfrac{-\text{Imag } f_{xy}(\omega)}{\text{Real } f_{xy}(\omega)}\right)$

$$= \arctan\left(\frac{-f_{xx}(\omega)\,\gamma_1\sin(\omega)}{f_{xx}(\omega)(\gamma_0 + \gamma_1\cos(\omega))}\right)$$

$$= \arctan\left(\frac{-\gamma_1\sin(\omega)}{\gamma_0 + \gamma_1\cos(\omega)}\right)$$

## 4. Multiple Spectral Estimators

### Sample Spectral Density

If $\underset{\sim}{X}(1), \ldots, \underset{\sim}{X}(T)$ is a sample from the CSTS $\underset{\sim}{X}(\cdot)$ we define the sample spectral density as

$$f_T(\omega) = \frac{1}{2\pi} \sum_{|v|<T} R_T(v)\, e^{-iv\omega}$$

$$= \frac{1}{2\pi}\left(\sum_{t=1}^{T} \underset{\sim}{X}(t)\, e^{it\omega}\right)\left(\sum_{t=1}^{T} \underset{\sim}{X}(t)\, e^{it\omega}\right)^*$$

where

$$R_T(v) = \begin{cases} \frac{1}{T}\sum_{t=1}^{T} \underset{\sim}{X}(t)\, \underset{\sim}{X}^T(t+v), & 0 \le v < T \\ R_T^{\,T}(-v), & -T < v \le 0 . \end{cases}$$

$R_T(\cdot)$ *is a consistent but correlated estimator of* $R(\cdot)$ while $f_T(\omega)$ is an asymptotically unbiased, but not a consistent, estimator of $f(\omega)$ in the sense that the variance of $f_T(\omega)$ is independent of $T$. Thus spectral estimation is concerned with finding estimators of $f(\omega)$ which are consistent.



### The Kernel Method

One approach to the spectral estimation problem is smoothing $f_T(\omega)$ via the kernel method. Here we use weighting functions (kernels) $K(\cdot)$ to form the filtered or smoothed sample spectral density:

$$f_{T,M}(\omega) = \int_{-\pi}^{\pi} K_M(\omega_0)\, f_T(\omega - \omega_0)\, d\omega_0$$

$$= \frac{1}{2\pi} \sum_{|v|\le M} k\left(\frac{v}{M}\right) R_T(V)\, e^{-iv\omega} ,$$

$$K_M(\omega) = \frac{1}{2\pi} \sum_{|v|\le M} k\left(\frac{v}{M}\right) e^{-iv\omega}$$

where $M$ is called the truncation point and

$$k(x) = \begin{cases} 1 & x = 0 \\ k(-x) & -1 \le x \le 1 \\ 0 & |x| > 1 \end{cases}$$

A kernel which performs well is the Parzen kernel (Parzen (1961)):

$$k(X) = \begin{cases} 1 - 6X^2 + 6|X|^3 , & |X| \le .5 \\ 2(1 - |X|)^3 , & .5 \le |X| \le 1 \\ 0 , & |X| > 1 \end{cases}$$

The Parzen estimator has a desirable property, that of positive definiteness. Since $f(\omega)$ is a positive definite function, we wish $f_{T,M}(\omega)$ to be.

The Kernel Method suffers from the fact that there is no objective way to choose $M$. Usually a range of $M$'s are used. As $M$ increases $f_{T,M}(\cdot)$ becomes wigglier. $\dot{f}_{T,M}(\omega) = \hat{J}(\omega)$

$$= \int_{-\pi}^{\pi} K_M(\omega_0) f_T(\omega - \omega_0)\, d\omega_0$$

estimates $J(\omega) = \int_{-\pi}^{\pi} K_M(\omega_0) f(\omega - \omega_0)\, d\omega_0$ which is used to approximate $f(\omega)$. Thus there are two sources of error,

i) $\hat{J}(\omega) - J(\omega)$ and

ii) $J(\omega) - f(\omega)$ .

The more points around $f_T(\omega)$ are used in the average, the closer $\hat{J}$ is to $J$ but the farther $J$ is from $f$ depending on the smoothness of $f$. Thus there is a tradeoff of variance and bias with no objective method of compromise.

The distribution of the kernel estimator is $V f_{T,M}(\omega)$ is d-dimensional complex Wishart with $V$ degrees of freedom and covariance $f(\omega)$ where

$$V^{-1} = \frac{M}{T} \int_{-\infty}^{\infty} k^2(u)\, du \; .$$



<u>Stationary Autoregressive Method</u>

Another approach to spectral estimation is to model the CSTS by an autoregressive model of order $p(AR(p))$ . New methods of order determination make this possible (Parzen (1974)).

The spectral estimator is

$$f_{\hat{p}}(\omega) = \frac{1}{2\pi} G_{\hat{p}}^{-1}(e^{i\omega}) \Sigma_{\hat{p}} G_{\hat{p}}^{-*}(e^{i\omega})$$

where

$$G_{\hat{p}}(z) = \sum_{j=0}^{\hat{p}} A_{\hat{p}}(j) z^j$$

and the $A_{\hat{p}}(j)$'s are solutions of the system $\sum_{j=0}^{\hat{p}} A_{\hat{p}}(j) R_T(j-v) = \delta_{v,0}\Sigma_{\hat{p}}$ , $v = 0,\ldots,\hat{p}$ , $\hat{p} = \min \mathrm{CAT}(m)$

$$\mathrm{CAT}(m) = \mathrm{Tr}\left[\frac{d}{T} \sum_{j=1}^{m} \hat{\hat{\Sigma}}_j^{-1} - \hat{\hat{\Sigma}}_m^{-1}\right], \quad m = 1,\ldots,M$$

$$\hat{\hat{\Sigma}}_j = \frac{T}{T - dj} \hat{\Sigma}_j$$

$$\hat{\Sigma}_j = \sum_{k=0}^{j} A_m(k) R_T(k)$$

There are two sources of error:

i) $\hat{G}_p - G_p$

ii) $G_p - G_\infty$

The CAT criterion $CAT(p)$ is a measure of the mean square error of approximating $G_\infty$ by $\hat{G}_p$. Thus CAT affords an objective method of compromising between variance and bias in the estimation.

**The Periodic Autoregressive Method**

Alternatively, one might use periodic autoregressive spectral estimators (Pagano (1976))

$$f_{\hat{p}}(\omega) = \frac{1}{2\pi} G_{\hat{p}}^{-1}(e^{i\omega}) \hat{\Sigma}_{\hat{p}} G_{\hat{p}}^{-*}(e^{i\omega})$$

where $G_{\hat{p}}(z) = \sum_{j=0}^{\hat{p}} A_{\hat{p}}(j) z^j$ and $\hat{p}$, $A_{\hat{p}}(\cdot)$, $\Sigma_{\hat{p}}$ are determined from $\hat{p}_1, \ldots, \hat{p}_d$, $\hat{\alpha}_k(j)$, $j = 1, \ldots, \hat{p}_k$, $k = 1, \ldots, d$ ; $\hat{\theta}_j^2$, $j = 1, \ldots, d$ found from

$$\sum_{j=0}^{\hat{p}_k} \hat{\alpha}_k(j) \hat{R}(k-j, k-v) = \delta_{v,0} \hat{\theta}_k^2 ,$$

$$v = 0, \ldots, \hat{p}_k \qquad k = 1, \ldots, d$$

$$\hat{R}(k, v) = \frac{1}{T} \sum_{j=0}^{T - \left[\frac{k+V}{d}\right]} Y(k + dj)\, Y(v + dj)$$

$$k = 1, \ldots, d \qquad v = 0, \ldots, Td - k + 1$$

$\hat{p}_k$ is chosen to minimize a mean square error type criterion, the PCAT criterion.



## 5. Examples and Conclusions

Simulations of sample size 200 were run for various values of the parameters $\alpha$, $\sigma_\epsilon^2$, $\sigma_\eta^2$, $\gamma_0$, and $\gamma_1$, where both white noise processes were normally distributed.

Four methods were used to estimate spectral quantities:

i) sample spectral density

ii) Parzen kernel estimator $(M = 60)$

iii) autoregressive model

iv) periodic autoregressive model.

Since the process is a first order multiple autoregression, the estimates of the AR order and parameters are of great interest.

The spectral quantities $f(\omega)$, $W_{yx}(\omega)$, $g_{yx}(\omega)$, $g_{xy}(\omega)$, and $\varphi_{xy}(\omega)$ are estimated and the estimates plotted over the interval $\omega \in (0, \pi)$ using the four methods of estimation. The plots were contrasted with plots of theoretical values.

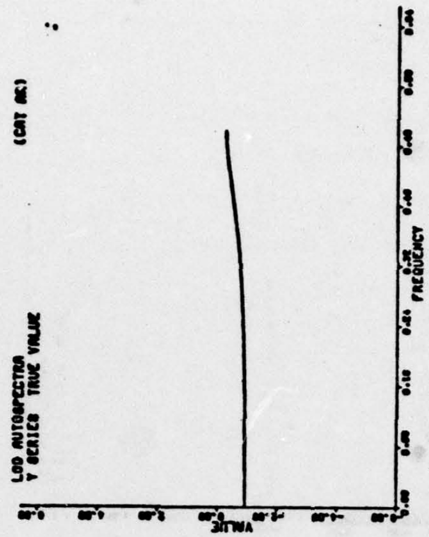
The conclusions formed from inspecting the plots from the various simulations are:

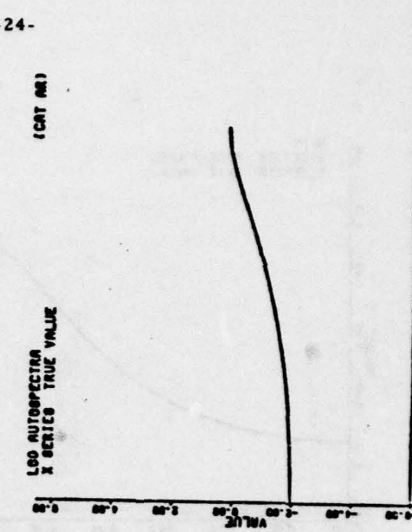
i) The two autoregressive methods performed about the same (possibly stationary AR slightly superior) and were clearly superior to the kernel estimator, which was clearly superior to the sample spectral density as we would expect.

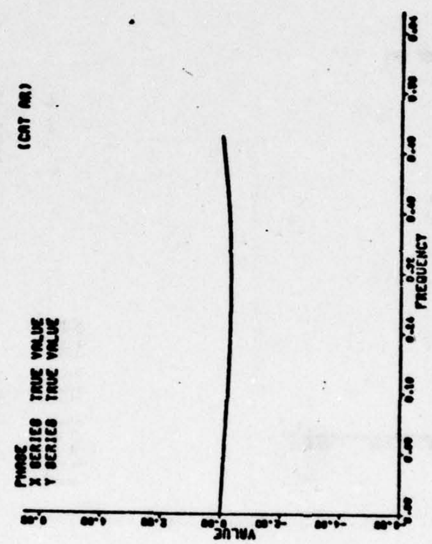
ii) The two autospectra and the phase were well estimated by the AR methods, while the estimates for coherency and gain often differed greatly from the true curves.

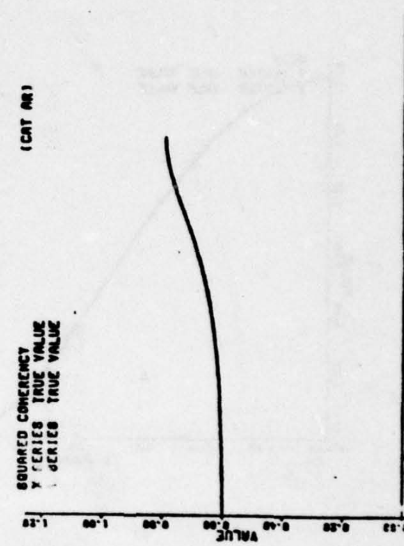
Following are the plots of the spectral estimates for our model with $\alpha = .5$, $\gamma_0 = 1.0$, $\gamma_1 = .3$, $\sigma_\epsilon^2 = 2.0$, $\sigma_\eta^2 = 1.0$. There were five samples of $N = 200$. The sample plots are preceded by plots of the true values.

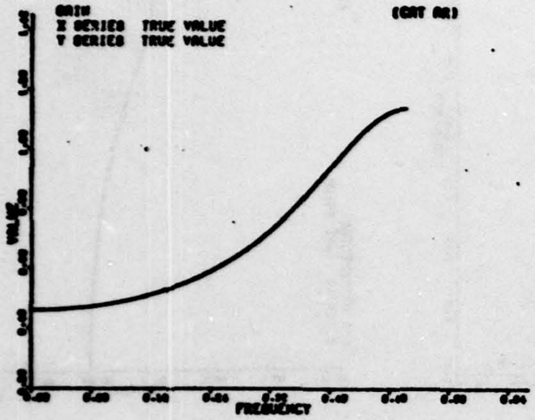


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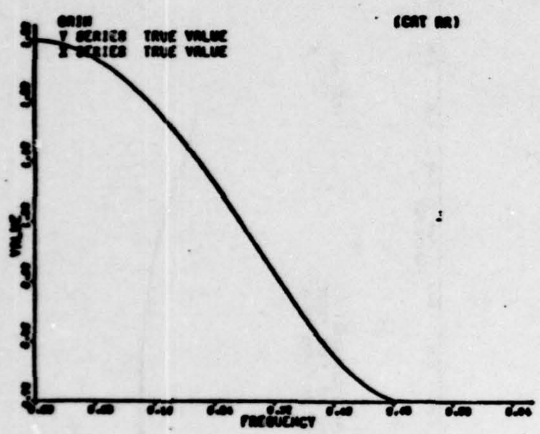
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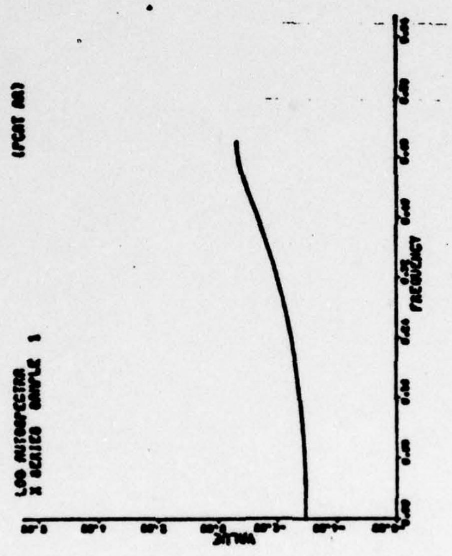
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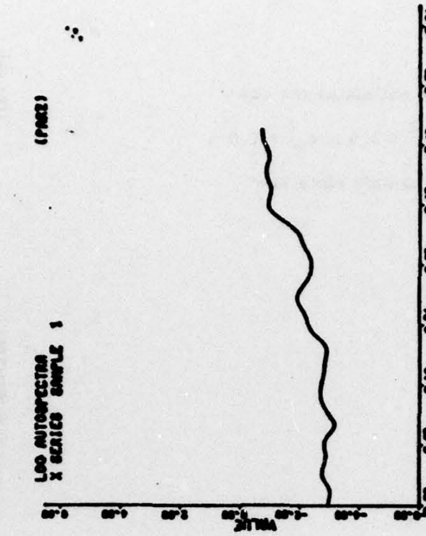
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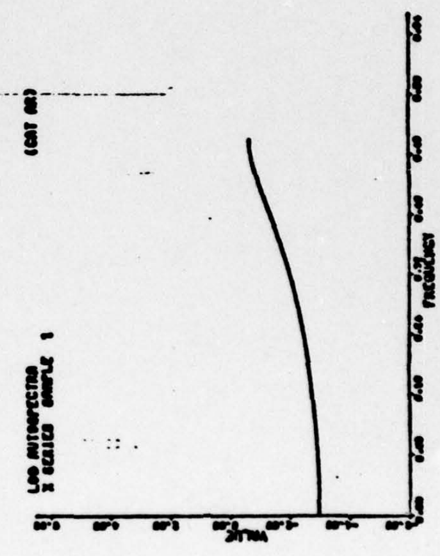
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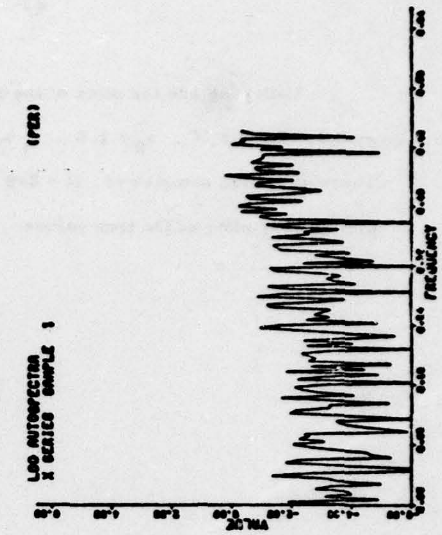
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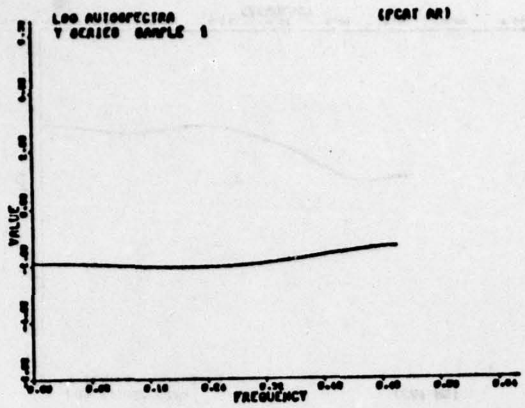
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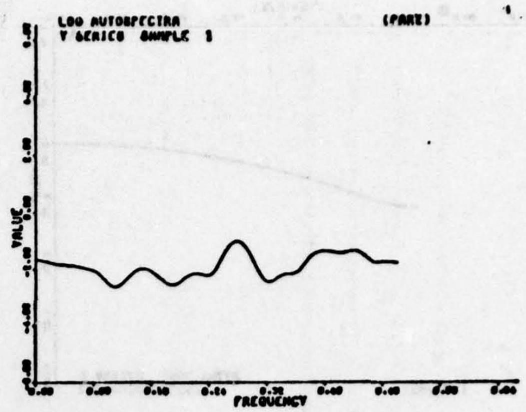
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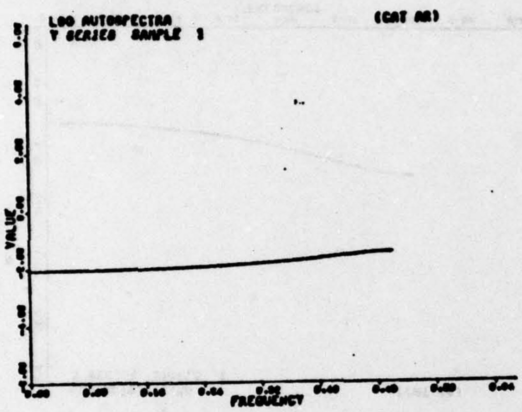
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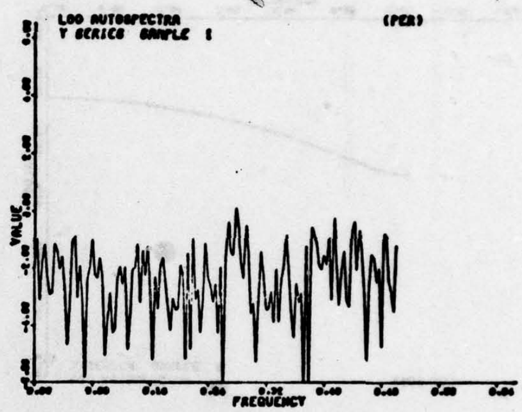
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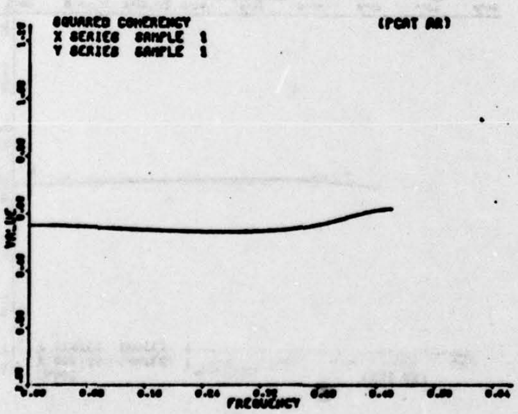
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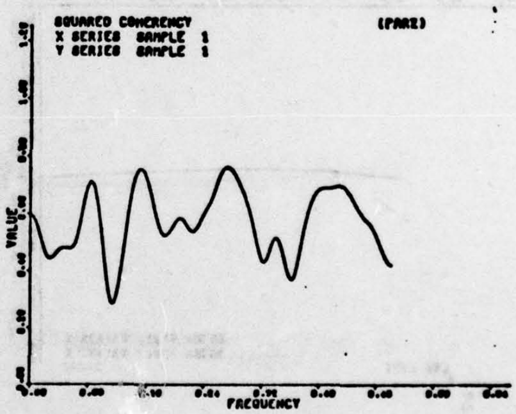
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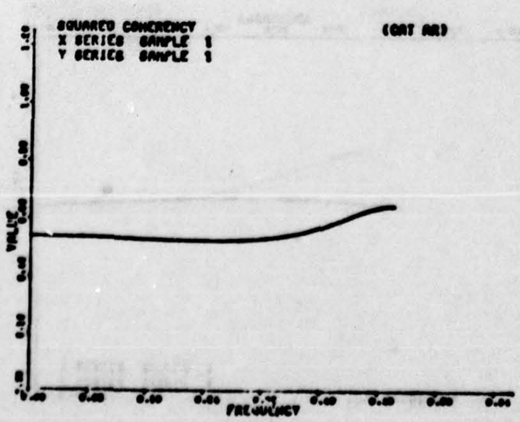
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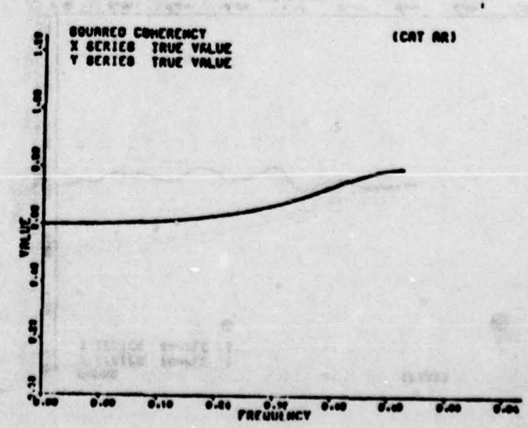
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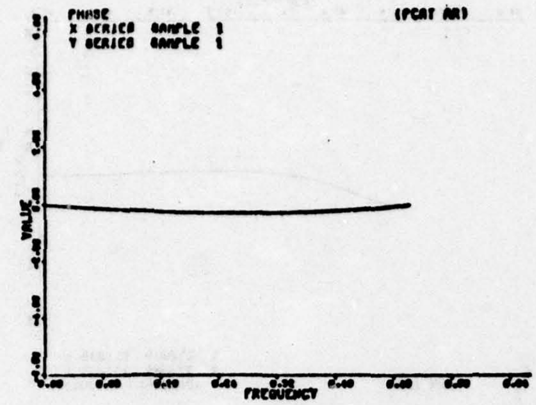
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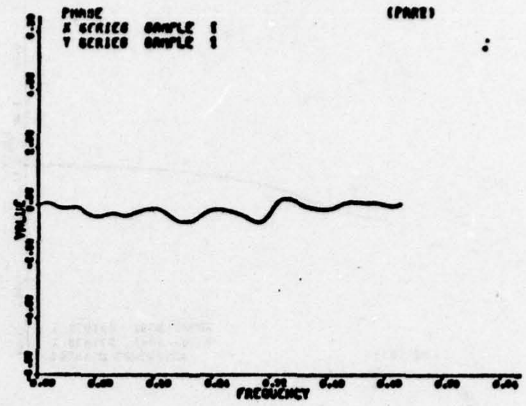
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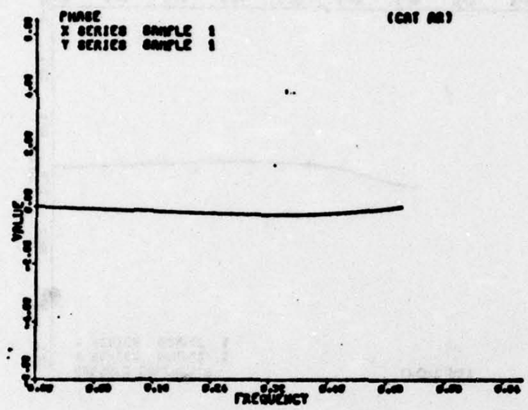
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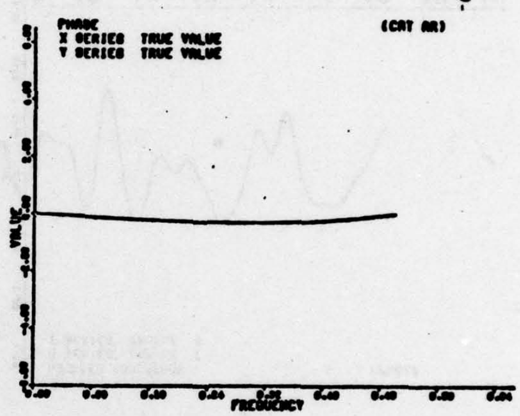
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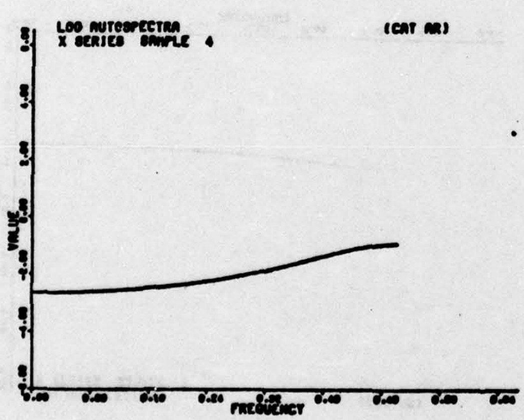
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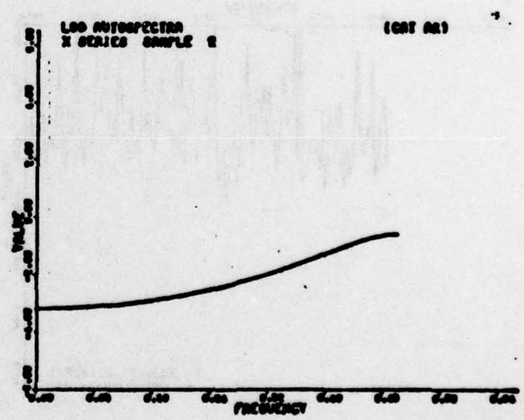
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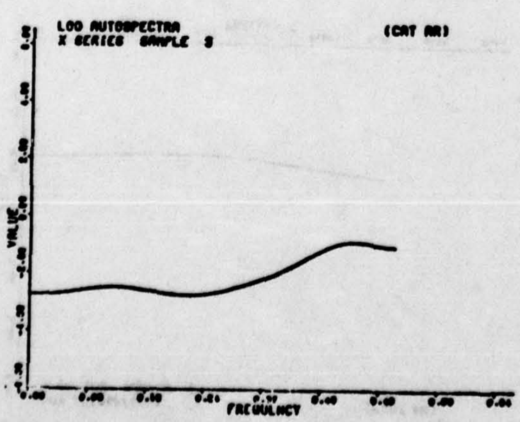
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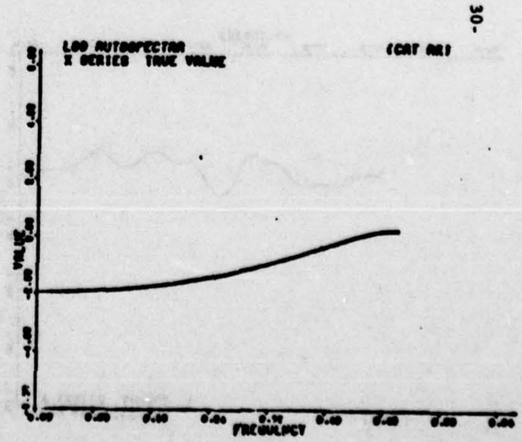
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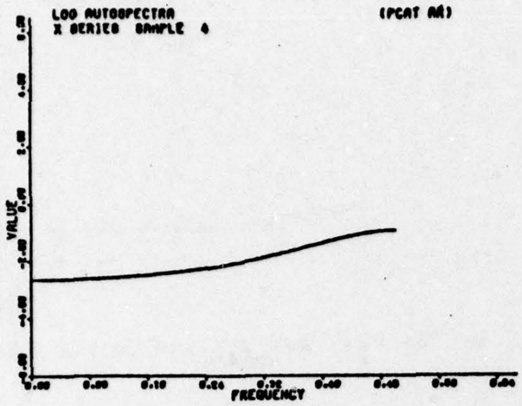
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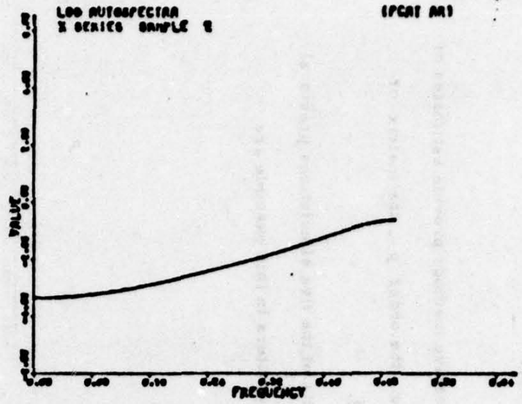
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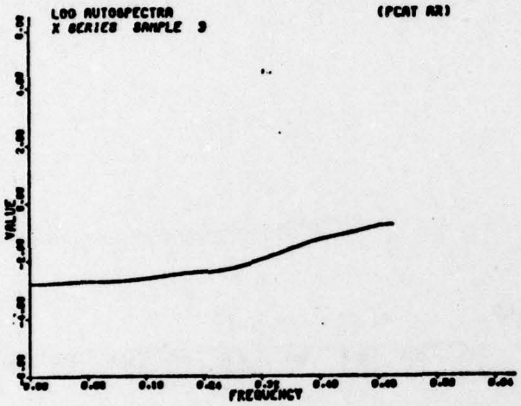
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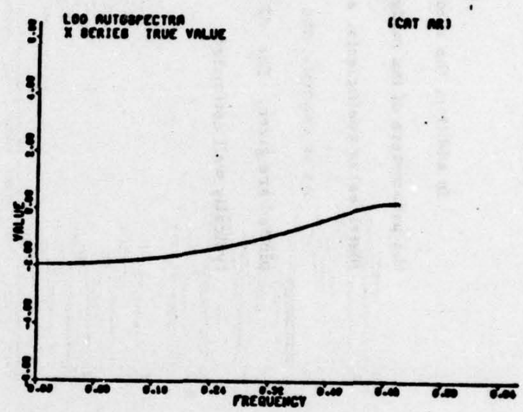
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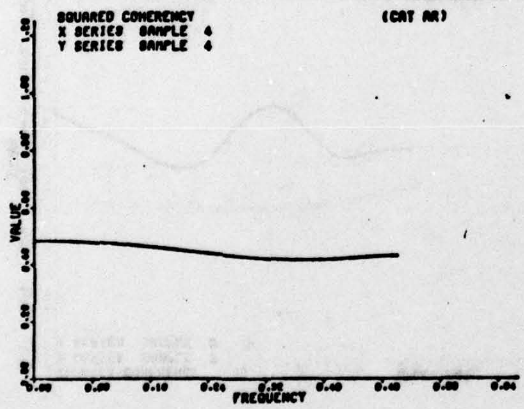
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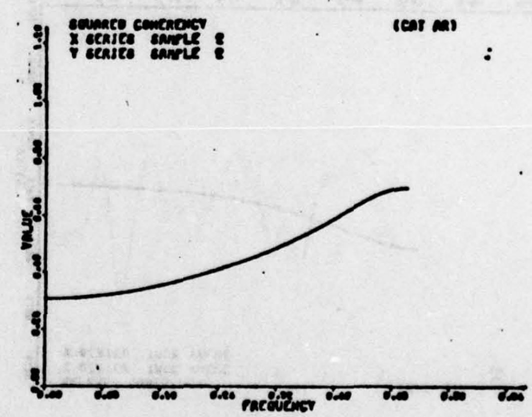
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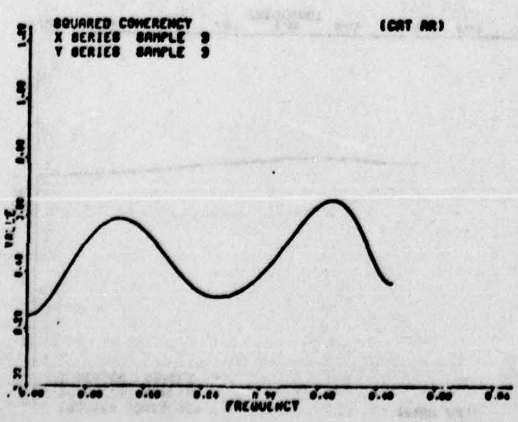
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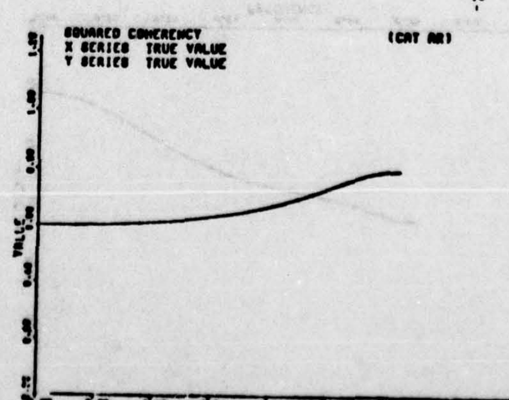
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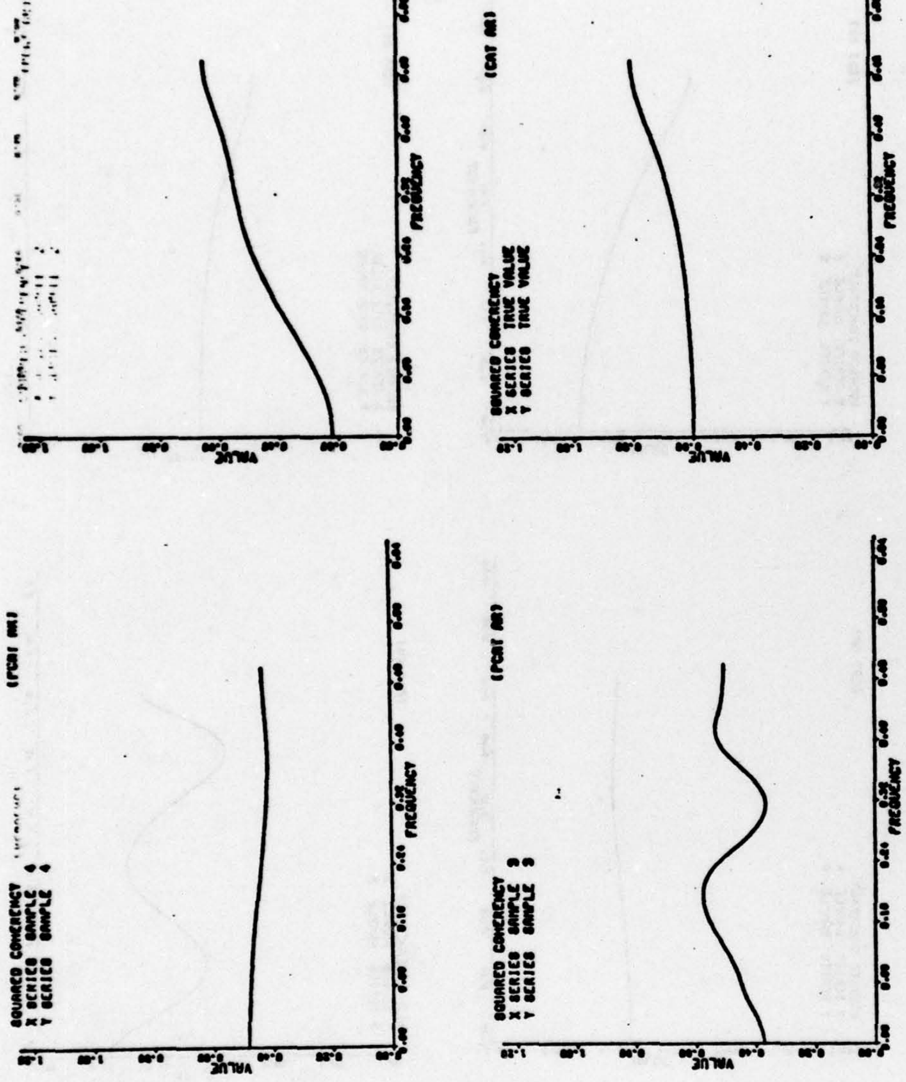




In addition, the autoregressive methods provide estimates of the parameters of the model, i.e. the order $p$, the matrix (or matrices) of coefficients, and $\Sigma$.

As an example, the results of the five simulations previously plotted are given. The AR parameters in this example are typically well estimated.

## AR

| Sample | Order | A(1) | $\hat{\Sigma}$ |
|---|---|---|---|
| True | 1 | $\begin{bmatrix} .5 & 0 \\ .2 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$ |
| 1 | 1 | $\begin{bmatrix} .57 & -.07 \\ .21 & .00 \end{bmatrix}$ | $\begin{bmatrix} 2 & 2.16 \\ 2.16 & 3.26 \end{bmatrix}$ |
| 2 | 1 | $\begin{bmatrix} .47 & .12 \\ .16 & .08 \end{bmatrix}$ | $\begin{bmatrix} 1.74 & 1.77 \\ 1.77 & 2.88 \end{bmatrix}$ |
| 3 | 4 | $\begin{bmatrix} .50 & -.11 \\ .19 & -.12 \end{bmatrix}$ | $\begin{bmatrix} 1.93 & 1.73 \\ 1.73 & 2.48 \end{bmatrix}$ |
| 4 | 1 | $\begin{bmatrix} .48 & -.08 \\ .12 & -.07 \end{bmatrix}$ | $\begin{bmatrix} 1.86 & 1.84 \\ 1.84 & 2.96 \end{bmatrix}$ |
| 5 | 2 | $\begin{bmatrix} .34 & .07 \\ .18 & .04 \end{bmatrix}$ | $\begin{bmatrix} 2.27 & 2.38 \\ 2.38 & 3.50 \end{bmatrix}$ |



## Periodic AR

| Sample | $P(P_1, P_2)$ | A(1) | $\Sigma$ |
|---|---|---|---|
| True | 1 (2, 3) | $\begin{bmatrix} .5 & 0 \\ .2 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$ |
| 1 | 2 (2, 4) | $\begin{bmatrix} .58 & -.07 \\ .14 & .03 \end{bmatrix}$ | $\begin{bmatrix} 2.05 & 2.17 \\ 2.17 & 3.28 \end{bmatrix}$ |
| 2 | 2 (3, 3) | $\begin{bmatrix} .56 & .08 \\ .24 & .04 \end{bmatrix}$ | $\begin{bmatrix} 1.71 & 1.74 \\ 1.74 & 2.86 \end{bmatrix}$ |
| 3 | 6 (2, 12) | $\begin{bmatrix} .61 & -.18 \\ .28 & -.19 \end{bmatrix}$ | $\begin{bmatrix} 2.00 & 1.80 \\ 1.80 & 2.50 \end{bmatrix}$ |
| 4 | 1 (2, 3) | $\begin{bmatrix} .48 & -.10 \\ .10 & -.07 \end{bmatrix}$ | $\begin{bmatrix} 1.86 & 1.84 \\ 1.84 & 2.96 \end{bmatrix}$ |
| 5 | 2 (2, 5) | $\begin{bmatrix} .45 & .02 \\ .27 & -.02 \end{bmatrix}$ | $\begin{bmatrix} 2.29 & 2.41 \\ 2.41 & 3.53 \end{bmatrix}$ |

**Recognizing the Model**

The best way to detect a model of this form is by autoregressive estimation. If order 1 is chosen, or higher orders with $A(j)$ $j > 1$ having all elements nearly zero, one may hypothesize an AR(1) model. Then if $A(1)$ is $\ni a_{12}$ and $a_{22}$ are nearly zero, the process can be well modeled by this model.

Then the parameters $\alpha$, $\gamma_0$, $\gamma_1$, $\sigma_\epsilon^2$, and $\sigma_\eta^2$ may be derived from $A$ and $\Sigma$.

$$A = \begin{bmatrix} \alpha & 0 \\ \alpha\gamma_0 - \gamma_1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_\epsilon^2 & \sigma_\epsilon^2\gamma_0 \\ \gamma_0\sigma_\epsilon^2 & \gamma_0^2\sigma_\epsilon^2 + \sigma_\eta^2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}.$$

Hence directly from $\hat{A}$ we have an estimate of $\alpha$ and directly from $\hat{\Sigma}$ we have an estimate of $\sigma_\epsilon^2$. Then

$$\gamma_0 = s_{21}/\sigma_\epsilon^2$$

$$\gamma_1 = -a_{21} + \alpha\gamma_0$$

and

$$\sigma_\eta^2 = s_{22} - \gamma_0^2\sigma_\epsilon^2$$



If one were just looking at plots, if $f_{xx}(\cdot)$, $f_{yy}(\cdot)$, $W_{yx}(\cdot)$, and $g_{yx}(\cdot)$ are monotonic one would suspect this to be an appropriate model and investigate further by estimating the AR parameters (if $W_{yx}(\cdot)$ and $g_{yx}(\cdot)$ were not monotonic, one should not necessarily disregard this model, since $W_{yx}(\cdot)$ and $g_{yx}(\cdot)$ are often poorly estimated). One would then look for the degenerate $2^{nd}$ column of $A$.

## References


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